Section 3-6, Mathematics 108

Rational Functions

Recall that a rational function has the form

$$f(x) = \frac{P(x)}{Q(x)}$$

Where P and Q are polynomials.

Also recall that wherever Q has a zero, the function will be undefined, so it's domain will not include this value.

The behavior of a rational function can differ from a polynomial function in a number of ways.

There are a number of tools we can use to determine what the behavior of the graph of the function looks like.

1) End behavior, what the function does when x gets very large or very negatively large.
2) Horizontal asymptotes
3) Vertical asymptotes
4) Zeros of the function, or X-intercepts
5) The Y-intercept = $f(0)$.

First let's consider the end behavior.

Three cases

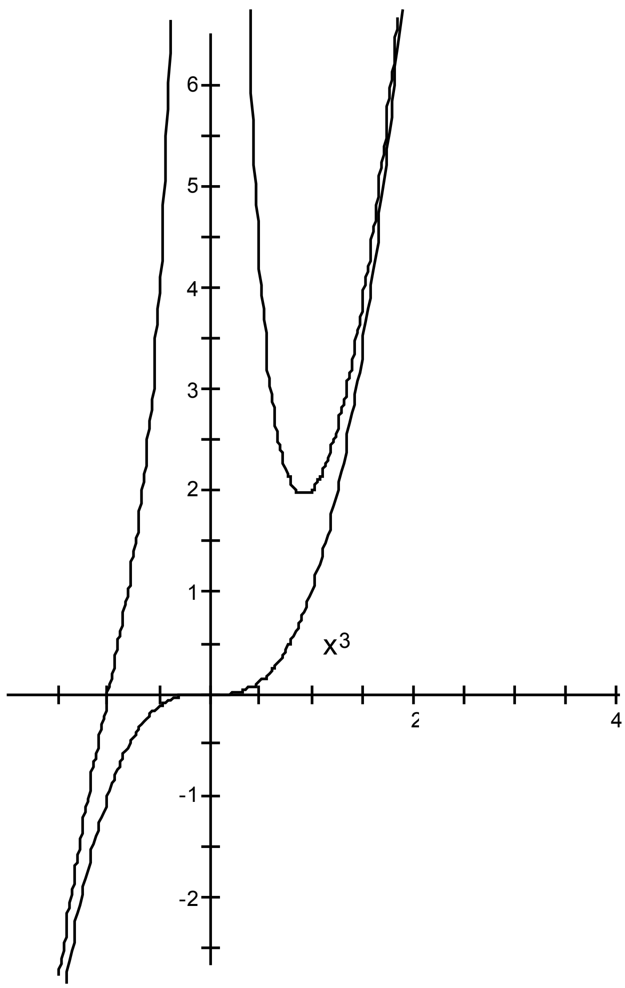
Case 1: The degree of P is greater than the degree of Q

Example

$$\frac{x^5+1}{x^2-1}$$

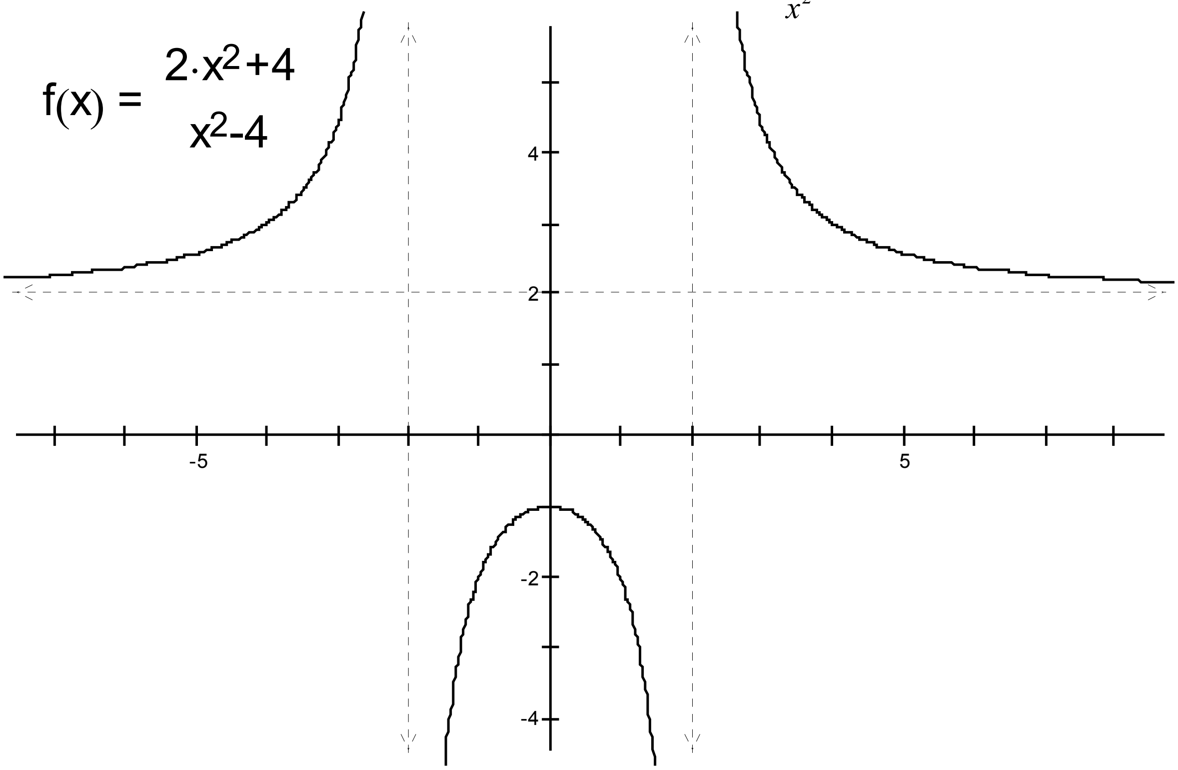
Note that for very large x or very negatively large x the 1 will become negligible and the function will approach $f(x)=x^3$

So it's end behavior will be the same.



Case 2: The degree of P is greater the same as the degree of Q
Example

In this case, as x gets very large the function will approach $\frac{2x^2}{x^2} = 2$



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The line $y=2$ that the function approaches in both directions is called an asymptote, specifically a horizontal asymptote.
An **asymptote** is a line that a function gets ever closer to but never reaches.

For horizontal asymptotes we can find out which side of the asymptote the function is on by putting in a large value, eg.

$$f(10) = \frac{2(10)^2 + 4}{(10)^2 - 4} = \frac{204}{96} \sim 2$$

$$f(-10) = \frac{2(-10)^2 + 4}{(-10)^2 - 4} = \frac{204}{96} \sim 2$$

Note that at $x = \pm 2$, $Q(x) = 0$. and we can see that there is a vertical asymptote.

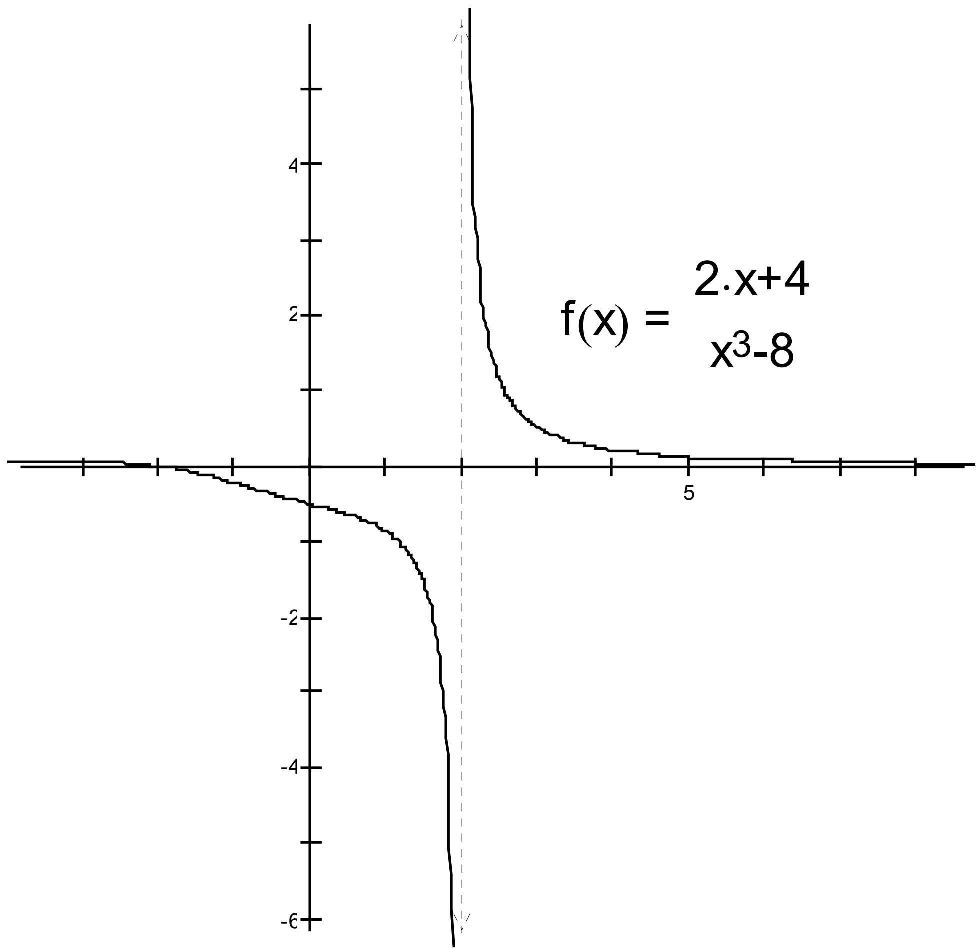
For vertical asymptotes you can see the behavior of the function by checking values on either side of the function,

$$f(2 - .01) = \frac{2(1.99)^2 + 4}{(1.99)^2 - 4} \sim -300 \quad f(2 + .01) = \frac{2(2.01)^2 + 4}{(2.01)^2 - 4} \sim 300$$

Case 3: The degree of P is less than the degree of Q

$$\frac{2x+4}{x^3-8}$$

In this case we expect the function to get small quickly, going to zero.

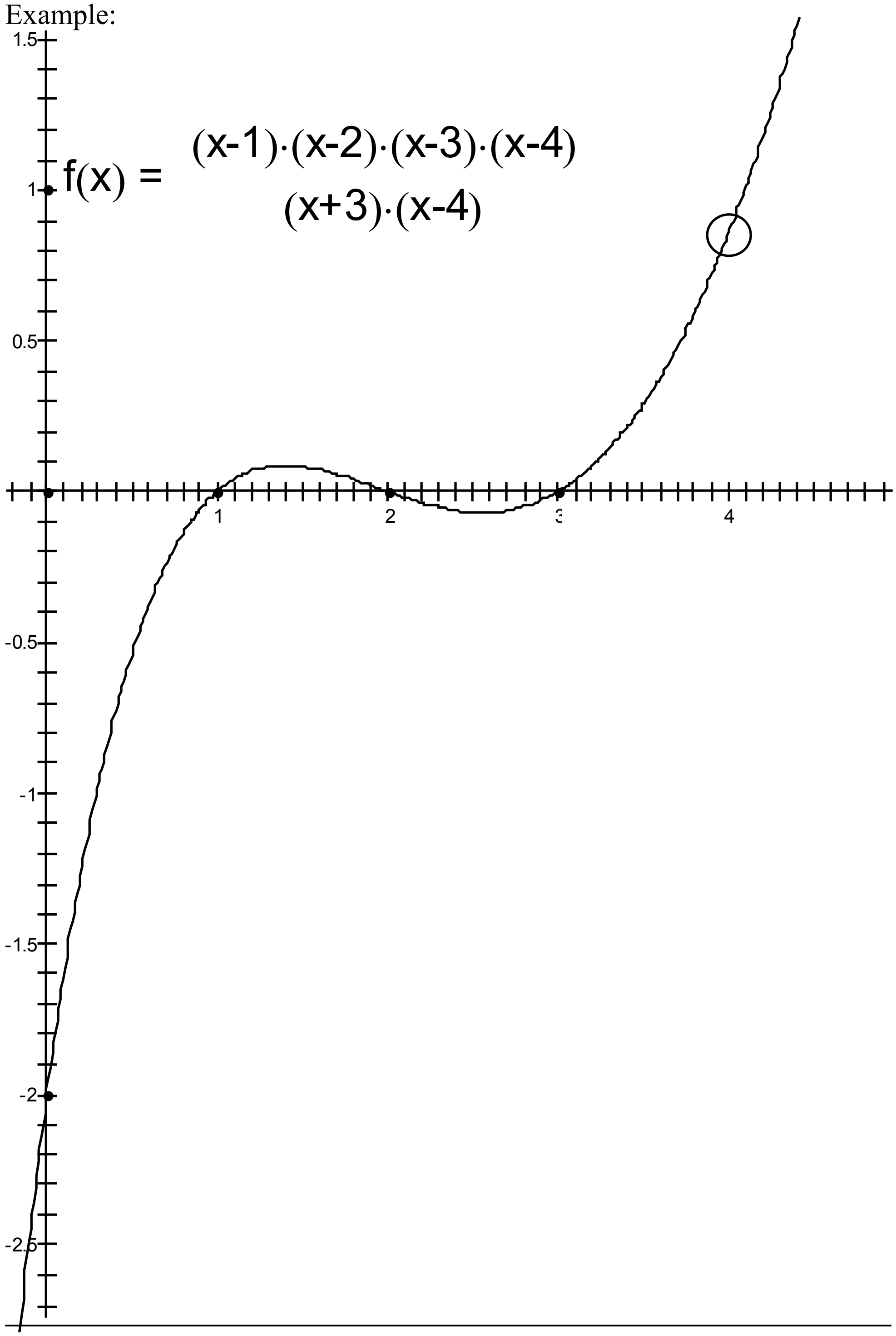


Notice the line at x=2 that the function descends near as it gets close to x=2 from the left and ascends as it gets close on the left. This is also an asymptote but a vertical one.

In general you will find that the graph of a rational function will have a vertical asymptote wherever $Q(x)$ has a zero.

Zero's of a rational function

Where ever $P(x)$ has a zero, the function will pass through the X-axis, unless $Q(x)$ also has a zero.
Where $y=P(0)$ the graph crosses the Y-axis
Example:



Canceling Common Factors

If *P* and *Q* have common factors, they may be canceled as long as you keep in mind any points where the function is not defined.

This is illustrated in the above example.

Example

$$f\left(x\right)=\frac{x-3}{x^2-3x}=\frac{x-3}{x\left(x-3\right)}=\frac{1}{x}$$

This will have the same graph as $\frac{1}{x}$ but the function will have a hole at x=3

Example

$$f\left(x\right)=\frac{x^3-2x^2}{x-2}=\frac{x^2\left(x-2\right)}{x-2}=x^2$$

This will have the same graph as x^2 but will have a hole at x=2

Transformations on Rational Functions

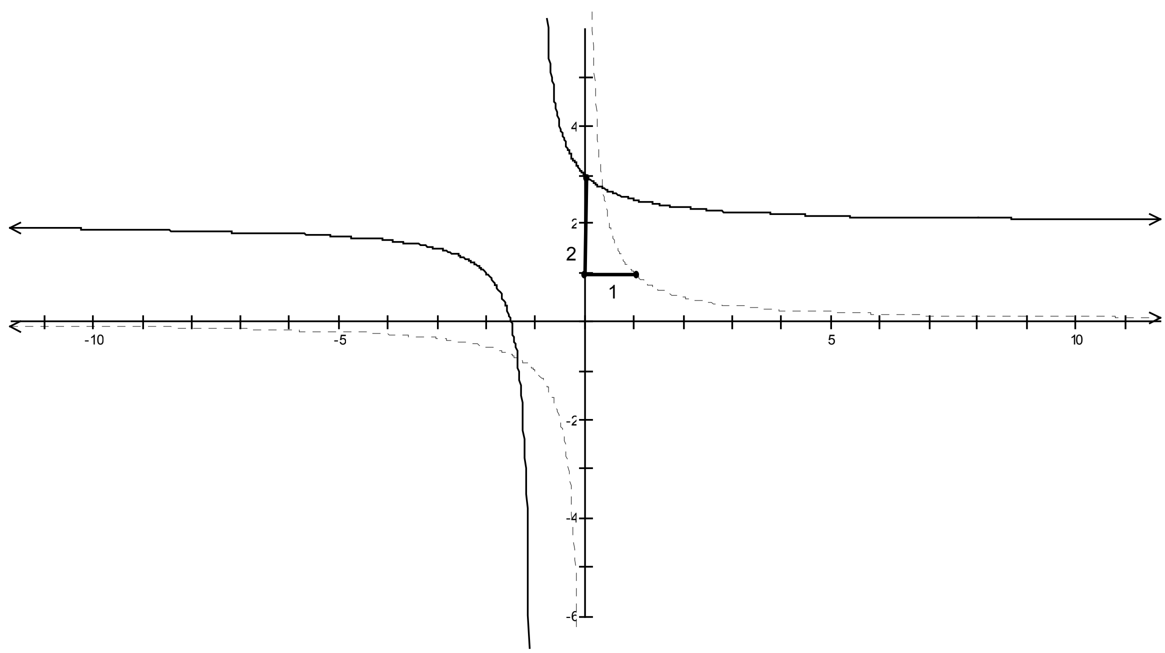
Just like any other function, we can transform a function using vertical and horizontal shifts.

Example

$$f(x) = \frac{2x+3}{x+1} = \frac{1}{x+1} + 2$$

This function is the same as $f(x) = \frac{1}{x}$

but shifted up 2 and to the left 1



Graphing a Rational Function

Graphing a rational function takes a little creativity.
Before attempting the graph it is good to gather some data.
Factor the numerator and denominator if you can.

1. What are the zeros of the denominator, where there will be vertical asymptotes?
2. What are the x and y intercepts?
3. What is the end behavior?
4. Does the function have any symmetry?

Example

$$f(x)=\frac{2x^2+7x-4}{x^2+x-2}=\frac{(2x-1)(x+4)}{(x-1)(x+2)}$$

We see immediately that the function has vertical asymptotes at 1 and -2

Setting x=0 we find $f(x)=\frac{-4}{-2}=2$ is the y intercept.

From the numerator we see that $y = 0$ at $x=\frac{1}{2}$ and $x=-4$

As the function gets very large it will approach $\frac{2x^2}{x^2}=2$

To see which direction it will approach 2 from we can put in some large values

$$f(-10)=\frac{200-70-4}{100-10-2}=\frac{126}{88}<2$$

so as x gets very negatively large it will approach 2 from below

$$f(10)=\frac{200+70-4}{100+10-2}=\frac{266}{108}>2$$

so as x gets very large it will approach 2 from above.

We can also check what happens near the vertical asymptotes by plugging in nearby points.

$$f(.75)=\frac{2(.75)^2+7(.75)-4}{(.75)^2+.75-2}=-3.45$$

$$f(1.25)=\frac{2(1.25)^2+7(1.25)-4}{(1.25)^2+1.25-2}=9.69$$

So on the left side of 1 the function is going negative toward the asymptote and on the right side it is going positive.

$$f(-2.25)=\frac{2(-2.25)^2+7(-2.25)-4}{(-2.25)^2+-2.25-2}=-11.84$$

$$f(-1.75)=\frac{2(-1.75)^2+7(-1.75)-4}{(-1.75)^2+-1.75-2}=14.7$$

So on the left side of -2 the function is going negative toward the asymptote and the right side it is going positive.

This gives us enough information to get a general feel for the graph of the function.